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Operator Transformations Between Exactly Solvable Potentials and Their Lie Group Generators

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Abstract

One may obtain, using operator transformations, algebraic relations between the Fourier transforms of the causal propagators of different exactly solvable potentials. These relations are derived for the shape invariant potentials. Also, potentials related by real transformation functions are shown to have the same spectrum generating algebra with Hermitian generators related by this operator transformation.

I. INTRODUCTION

The study of exactly solvable potentials, for which the quantum mechanical eigenfunctions may be expressed in terms of hypergeometric functions, has a long and varied history. One approach is an algebraic solution of the problem. Early work by Infeld and Hull classified factorizations of the Schroedinger operator for solvable potentials which then allow one to generate other solutions to the problem. [1] A related technique, supersymmetric quantum mechanics, discovered as a limiting case ($d = 1$) of supersymmetric field theory, was

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introduced by Witten and later developed by other authors. [2] In particular, Gendenshtein gave a criteria, shape invariance, which when satified insures that the complete spectrum of the supersymmetric Hamiltonian may be found. [3] Finally, spectrum generating algebras, whose use dates back to Pauli's work on the hydrogen atom, have been studied more recently as a method to find the spectrum and eigenstates of solvable potentials. [4,5]

Another method to find the energy eigenvalues and wavefunctions of a solvable potential is to use an operator transformation, essentially a change of independent and dependent variables, to relate it to a Schroedinger equation for a potential whose solutions are known. Duru and Kleinert described such a method for transforming the resolvant operator, whose matrix element is the propagator. [6] They used this technique to transform the time-sliced form of a path integral into a known path integral, such as that for the harmonic oscillator, by transforming both the space and time variables in the path integral expression. We will discuss these transformations, outside the context of path integrals, in the next section.

We will show that the operator transformations not only allow one to find algebraic relations between the Fourier transform of the propagators for two different quantum systems but also, in the case of a real transformation function, provide a mapping between the group generators for the spectrum generating algebra. Thus quantum systems which may be mapped to one another by real Duru-Kleinert transformations have the same formulation in terms of the enveloping algebra of the same Lie group.

In the first section we describe the operator transformations with special attention to how the measure for the normalization of states transforms. We next illustrate the method with a derivation of the relation between the propagators for the trigonometric Poschl-Teller and Rosen-Morse potentials and give the relations for the propagators for some other exactly solvable potentials. Finally we examine the corresponding transformation of the Lie group generators.

II. OPERATOR TRANSFORMATIONS AND CAUSAL GREEN'S FUNCTIONS

We will consider transformations of the Fourier transform of the causal propagator for a quantum mechanical system. Hereafter operators will be denoted by a caret. The propagator is given by

$$K(x_0, x_f, t) \equiv \theta(t) \, \langle x_f | e^{-\frac{\imath}{\hbar}\hat{\mathcal{H}}t} | x_0 \rangle \tag{1}$$

and its Fourier transform is defined by

$$\begin{aligned} G(x_0, x_f, E) &\equiv \imath \int_{-\infty}^{\infty} dt \, e^{\frac{\imath}{\hbar}Et} \, K(x_0, x_f, t) \\ &= \imath \int_0^{\infty} dt \, \langle x_f | e^{-\frac{\imath}{\hbar}(\hat{\mathcal{H}}-E)t} | x_0 \rangle \\ &= \langle x_f | \frac{\hbar}{\hat{\mathcal{H}} - E - \imath\epsilon} | x_0 \rangle \end{aligned} \tag{2}$$

where the infinitesimal imaginary constant in the last line gives the causal propagator.

Duru and Kleinert realized that Eqn. 2 is invariant under two types of operator transformations. One type is simply a point canonical transformation, which for a one-dimensional system is

$$\begin{aligned} \hat{x} &\rightarrow f(\hat{x}) \\ \hat{p} &\rightarrow \frac{1}{f'(\hat{x})}\hat{p} \end{aligned} \tag{3}$$

with $\hat{x}$, $\hat{p}$ the canonical position and momentum respectively. This point canonical transformation may be implemented by a similarity transformation on the operators, which is also called a quantum canonical transformation, since if it is applied to all operators it preserves the canonical commutation relations. [7] Under such a similarity transformation

$$\hat{\mathcal{H}} - E \rightarrow \hat{\mathcal{O}}(\hat{\mathcal{H}} - E)\hat{\mathcal{O}}^{-1} \tag{4}$$

$$\langle x| \rightarrow \langle x|\hat{\mathcal{O}}^{-1}. \tag{5}$$

The operator $\hat{\mathcal{O}}$ which implements the transformation is composed of the canonical position and momentum operators. We will assume that $\hat{\mathcal{O}}$ is invertible although, with proper care,

operators with a nonzero kernel may also be considered. [7] Clearly, this type of transformation leaves invariant any matrix element of an operator.

Another type of transformation which leaves Eqn. 3 invariant is what Duru and Kleinert denoted as an f-transformation. We will distinguish between two types of f-transformations, since the normalization measure transforms differently in each case. The first type of f-transformation is a similarity transformation with $\hat{\mathcal{O}} = f(\hat{x})$, where $f(q)$ is some function of q. The other type of transformation, which we will call conjugation, is

$$\hat{\mathcal{H}} - E \to f(\hat{x})(\hat{\mathcal{H}} - E)f(\hat{x}) \tag{6}$$

$$\langle x| \to \langle x|f(\hat{x}). \tag{7}$$

Eqn. 3 is invariant under this transformation, however a general matrix element of an operator is not invariant.

We next examine the change in the measure factor for these transformations. First consider a similarity transformation, Eqn. 5. The original wavefunction $\psi(r)$ and the transformed wavefunction $\psi'(r)$ are defined as

$$\psi(r) = \langle r|\psi\rangle \tag{8}$$

$$\psi'(r) = \langle r|\hat{\mathcal{O}}|\psi\rangle \tag{9}$$

with $\langle r|$ an eigenstate of the position operator with eigenvalue r. We then may find the transformation of the (in general operator valued) measure factor $\hat{\mu}$.

$$\begin{aligned}
\langle\psi|\psi\rangle_{\hat{\mu}} &= \int dr \langle\psi|\hat{\mu}|r\rangle\langle r|\psi\rangle \\
&= \int dr \langle\psi|\hat{\mu}\hat{\mathcal{O}}^{-1}|r\rangle\langle r|\hat{\mathcal{O}}|\psi\rangle \\
&= \int dr \langle\psi|\hat{\mathcal{O}}^{\dagger}(\hat{\mathcal{O}}^{-1})^{\dagger}\hat{\mu}\hat{\mathcal{O}}^{-1}|r\rangle\langle r|\hat{\mathcal{O}}|\psi\rangle \\
&= \int dr \langle\psi|\hat{\mathcal{O}}^{\dagger}\hat{\mu}'|r\rangle\langle r|\hat{\mathcal{O}}|\psi\rangle \\
&= \langle\psi'|\psi'\rangle_{\hat{\mu}'}.
\end{aligned} \tag{10}$$

Therefore the measure factor for the transformed wavefunctions is $\hat{\mu}' = (\hat{\mathcal{O}}^{-1})^{\dagger}\hat{\mu}\hat{\mathcal{O}}^{-1}$.

We next assume that the measure factor contains only the position operator, *i.e.*, $\hat{\mu} = g(\hat{x})$. Without ambiguity we may then use the notation $g(r)$ for the measure factor. For a point canonical transformation, Eqn. 4, the measure transforms as a differential

$$g(r) \rightarrow g(f(r))\frac{df(r)}{dr}. \tag{11}$$

For the similarity transformation with $\hat{\mathcal{O}} = f(\hat{x})$ the measure factor transforms multiplicatively as

$$g(r) \rightarrow f^{-2}(r)g(r). \tag{12}$$

Finally the measure factor remains unchanged for the conjugation transformation of Eqn. 7.

III. EXAMPLE: ROSEN-MORSE TO POSCHL-TELLER POTENTIAL

The transformation from a Hamiltonian with potential $V_0(r)$

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2\mu} + V_0(\hat{x}) \tag{13}$$

to another with potential $V_f(r)$ is specified by a single function $f(r)$. We will illustrate the general sequence of transformations along with the specific example with $V_0(r)$ the Rosen-Morse I potential and $V_f(r)$ the hyperbolic Poschl-Teller potential.

First a point canonical transformed is performed as in Eqn. 4. For the example, the function is $f(r) = \frac{1}{a}\,\mathrm{arctanh}\cos 2\mathrm{ar}$, giving the operator transformation

$$\begin{aligned} \hat{x} &\rightarrow \hat{\mathcal{O}}_0\hat{x}\hat{\mathcal{O}}_0^{-1} = \frac{1}{a}\,\mathrm{arctanh}\cos 2\mathrm{a}\hat{\mathrm{x}} \\ \hat{p} &\rightarrow \hat{\mathcal{O}}_0\hat{p}\hat{\mathcal{O}}_0^{-1} = -\frac{1}{2}\left(\sin 2a\hat{x}\right)\hat{p}, \end{aligned} \tag{14}$$

which transforms the original operator, $\hat{\mathcal{S}}_0 \equiv \hat{\mathcal{H}}_0 - E$

$$\hat{\mathcal{S}}_0 = \frac{1}{2\mu}\hat{p}^2 + A\ \mathrm{csch}^2\mathrm{a}\hat{\mathrm{x}} + \mathrm{B}\coth\mathrm{ax}\ \mathrm{cschax} - \mathrm{E} \tag{15}$$

into

$$\hat{\mathcal{S}}_1 \equiv \hat{\mathcal{O}}_0 \hat{\mathcal{S}}_0 \hat{\mathcal{O}}_0^{-1} = \frac{1}{8\mu}\left(\sin^2 2a\hat{x}\hat{p}^2 - 2a\imath\hbar \sin 2a\hat{x}\cos 2ax\hat{p}\right) + A\cos 2a\hat{x} - B\sin^2 2a\hat{x} - E. \tag{16}$$

According to Eqn. 11 the measure transforms as

$$dx \to \frac{-2}{\sin 2a\hat{x}} dx. \tag{17}$$

The propagator becomes

$$G_{\mathrm{R-M}}(x_f, x_0, E) = \imath \int dT \langle x_f |\, e^{-\frac{\imath}{\hbar}\hat{\mathcal{S}}_0 T}\, | x_0 \rangle \tag{18}$$

$$= \imath \int dT \langle x_f |\, \hat{\mathcal{O}}_0^{-1} e^{-\frac{\imath}{\hbar}\hat{\mathcal{S}}_0 T} \left(\hat{\mathcal{O}}^{-1}\right)^{\dagger}\, | x_0 \rangle \tag{19}$$

$$= \imath \int dT \langle \frac{1}{2a}\arccos(\tanh ax_f) |\, e^{-\frac{\imath}{\hbar}\hat{\mathcal{S}}_1 T}\, | \frac{1}{2a}\arccos(\tanh ax_0) \rangle.$$

Next one performs the similarity transformation with $\hat{\mathcal{O}}_1 = \left(\frac{df(r)}{dr}\right)^{\frac{1}{2}} = \sin^{-\frac{1}{2}} 2a\hat{x}$ to get

$$\hat{\mathcal{S}}_2 \equiv \hat{\mathcal{O}}_1 \hat{\mathcal{S}}_1 \hat{\mathcal{O}}_1^{-1} = \frac{1}{8\mu}\sin^2 2a\hat{x}\hat{p} - \frac{\imath\hbar a}{2\mu}\sin 2a\hat{x}\cos 2a\hat{x}\hat{p} + \frac{2\hbar^2 a^2}{8\mu}\sin^2 2a\hat{x} \tag{20}$$

$$+ A\cos 2a\hat{x} - B\sin^2 2a\hat{x} - \frac{\hbar^2 a^2}{8\mu} - E.$$

The measure transforms as

$$\frac{-2}{\sin 2a\hat{x}} dx \to -2dx \tag{21}$$

and the propagator is then

$$G_{\mathrm{R-M}}\ (\ x_f, x_0, E) = \imath \int dT \langle \frac{1}{2a}\arccos(\tanh ax_f) |\, \hat{\mathcal{O}}^{-1} e^{-\frac{\imath}{\hbar}\hat{\mathcal{S}}_2 T} \left(\hat{\mathcal{O}}^{-1}\right)^{\dagger}\, | \frac{1}{2a}\arccos(\tanh ax_0) \rangle \tag{22}$$

$$= \imath(\ \mathrm{sech}^{\frac{1}{2}} \mathrm{ax_f})(\ \mathrm{sech}^{\frac{1}{2}} \mathrm{ax_0}) \int \mathrm{dT} \langle \frac{1}{2\mathrm{a}}\arccos(\tanh \mathrm{ax_f}) |\, \mathrm{e}^{-\frac{\imath}{\hbar}\hat{\mathcal{S}}_2 \mathrm{T}}\, | \frac{1}{2\mathrm{a}}\arccos(\tanh \mathrm{ax_0}) \rangle.$$

Next a conjugation transformation follows, Eqn. 7, with the function $C\frac{df(r)}{dr}$. The constant C is chosen to give the correct kinetic energy factor in the Hamiltonian.

$$\hat{\mathcal{S}}_3 \equiv \frac{2}{\sin 2a\hat{x}} \hat{\mathcal{S}}_2 \frac{2}{\sin 2a\hat{x}} = \frac{1}{2\mu}\hat{p}^2 + \left(A - E - \frac{\hbar^2 a^2}{8\mu}\right)\csc^2 2a\hat{x} \tag{23}$$

$$+ \left(-A - E - \frac{\hbar^2 a^2}{8\mu}\right)\sec^2 2a\hat{x} - \frac{1}{2}\hbar^2 a^2 - 4B.$$

The transformed propagator is

$$
\begin{aligned}
G_{\mathrm{R-M}}\ \ (\ x_f, x_0, E) &= \imath(\ \mathrm{sech}^{\frac{1}{2}}\mathrm{ax_f})(\ \mathrm{sech}^{\frac{1}{2}}\mathrm{ax_0}) \\
&\times \int dT \langle \frac{1}{2a}\arccos(\tanh ax_f)|\ \left(\frac{2}{\sin 2a\hat{x}}\right) e^{-\frac{\imath}{\hbar}\hat{\mathcal{S}}_3 T} \left(\frac{2}{\sin 2a\hat{x}}\right)\ |\frac{1}{2a}\arccos(\tanh ax_0)\rangle \\
&= 4\imath(\cosh^{\frac{1}{2}} ax_f(\cosh^{\frac{1}{2}} ax_0) \\
&\times \int dT \langle \frac{1}{2a}\arccos(\tanh ax_f)|\, e^{-\frac{\imath}{\hbar}\hat{\mathcal{S}}_3 T}\ |\frac{1}{2a}\arccos(\tanh ax_f)\rangle.
\end{aligned} \tag{24}
$$

Finally the the Hilbert space is rescaled so that the measure becomes the usual one, $\mu = dx$,

$$
{}^{\mathrm{norm}}\langle x| \equiv \sqrt{2}\langle x|. \tag{25}
$$

This introduces a factor of $\frac{1}{2}$ in the propagator, Eqn. 25. The final result is then obtained from Eqn. 25 by matching parameters in the operator $\hat{\mathcal{S}}_3$ with those for the Poschl-Teller potential. The algebraic relations between the Fourier transform of the propagator for several solvable potentials are shown in the table along with the function $f(r)$ used for the operator transformations. [1] Although all of the potentials for which we give explicit results in the table are shape invariant, the operator transformations are valid for a general potential. It is interesting to note that although not all one dimensional solvable potentials, classified by Natanzon, are shape invariant, they are related to a shape invariant potential by an operator transformation. [9,10]

IV. OPERATOR TRANSFORMATIONS FOR LIE GROUP GENERATORS

The operator transformations from $\hat{\mathcal{S}}_0 \equiv \hat{\mathcal{H}}_0 - E_0$ to $\hat{\mathcal{S}}_f \equiv \hat{\mathcal{H}}_f - E_f$ may be summarized by

$$
\hat{\mathcal{S}}_f = C\ (f')^{3/2}\ \hat{\mathcal{O}}_0\, \hat{\mathcal{S}}_0\, \hat{\mathcal{O}}_0^{-1}\ (f')^{\frac{1}{2}}. \tag{26}
$$

---

[1] The transformation functions given in the table are also listed in Ref. [8], however we correct them for the Rosen-Morse II and Eckart potentials.

$\hat{\mathcal{O}}_0$ is the operator implementing the point canonical transformation, Eqn. 4, with function $f(q)$ and C is a constant. Since the eigenvalue equation, $\hat{\mathcal{S}}_f = 0$, is homogeneous one may multiply Eqn. 26 by $C^{-1}(f')^{-2}$ on the left to obtain the following equation, valid for an interval in which $f' \neq 0$ and finite,

$$(f')^{-\frac{1}{2}}\, \hat{\mathcal{O}}_0\, \hat{\mathcal{S}}_0\, \hat{\mathcal{O}}_0^{-1}\, (f')^{\frac{1}{2}} = 0. \tag{27}$$

The operator transformation between the eigenvalue equation for the Hamiltonian $\hat{\mathcal{H}}_0$ and $\hat{\mathcal{H}}_f$ now preserves the commutators of operators on the two Hilbert spaces, *e.g.*, it is a Lie algebra isomorphism. The new generators $\hat{T}^i_f$ are related to the Lie algebra generators for the original potential, $\hat{T}^i_0$ as

$$\hat{T}^i_f = (f')^{-\frac{1}{2}}\, \hat{\mathcal{O}}_0\, \hat{T}^i_0\, \hat{\mathcal{O}}_0^{-1}\, (f')^{\frac{1}{2}} \tag{28}$$

Therefore, in the cases where the eigenvalue equation for $\hat{\mathcal{H}}_0$ may be written as an element of the enveloping algebra of a particular Lie algebra, the transformed eigenvalue equation, Eqn. 27, has the same formulation in terms of Lie group generators, however in a different representation. The eigenvalue equation for the potentials listed in the table then have the same Lie algebraic form as either the radial harmonic oscillator, the trigonometric Poschl-Teller, or the hyperbolic Poschl-Teller potential. $SU(1,1)$ generators for the radial harmonic oscillator Schroedinger operator and those related to it by Eqn. 28 are well known and given in Ref. [11].

As an example, we consider the Lie algebraic form for the trigonometric Poschl-Teller potential and then find the transformed generators for the Rosen-Morse I potential. The Poschl-Teller potential is known to have an algebraic formulation in terms of the Lie group $SU(2) \otimes SU(2)$. One may find the generators for $SO(4) = SU(2) \otimes SU(2)$ by considering the generators of rotations in $\Re^4$

$$\begin{aligned} J_1 &= \frac{\imath}{2}\left(-x_1\partial_4 + x_2\partial_3 - x_3\partial_2 + x_4\partial_1\right), \\ J_2 &= \frac{\imath}{2}\left(-x_1\partial_3 - x_2\partial_4 + x_3\partial_1 + x_4\partial_2\right), \end{aligned} \tag{29}$$

$$
\begin{aligned}
J_3 &= \frac{\imath}{2}\left(-x_1\partial_2 + x_2\partial_1 + x_3\partial_4 - x_4\partial_3\right), \\
K_1 &= \frac{\imath}{2}\left(-x_1\partial_2 + x_2\partial_1 - x_3\partial_4 + x_4\partial_3\right), \\
K_2 &= \frac{\imath}{2}\left(x_1\partial_3 - x_2\partial_4 - x_3\partial_1 + x_4\partial_2\right), \\
K_3 &= \frac{\imath}{2}\left(x_1\partial_4 + x_2\partial_3 - x_3\partial_2 - x_4\partial_1\right).
\end{aligned}
$$

Changing to Euler angle coordinates for the double cover of S^3

$$
\begin{aligned}
x_1 &= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi+\psi}{2}\right), \\
x_2 &= \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi+\psi}{2}\right), \\
x_3 &= \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi-\psi}{2}\right), \\
x_4 &= \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi-\psi}{2}\right),
\end{aligned}
\tag{30}
$$

and scaling $\theta \rightarrow 2a\theta$ we obtain the generators

$$
\begin{aligned}
J_1 &= \imath\left(\frac{1}{2a}\sin\psi\partial_\theta - \csc 2a\theta\cos\psi\partial_\phi + \cot 2a\theta\cos\psi\partial_\psi\right), \\
J_2 &= \imath\left(-\frac{1}{2a}\cos\psi\partial_\theta - \csc 2a\theta\sin\psi\partial_\phi + \cot 2a\theta\sin\psi\partial_\psi\right), \\
J_3 &= -\imath\partial_\psi, \\
K_1 &= \imath\left(\frac{1}{2a}\sin\phi\partial_\theta + \cot 2a\theta\cos\phi\partial_\phi - \csc 2a\theta\cos\phi\partial_\psi\right), \\
K_2 &= \imath\left(-\frac{1}{2a}\cos\phi\partial_\theta + \cot 2a\theta\sin\phi\partial_\phi - \csc 2a\theta\sin\phi\partial_\psi\right), \\
K_3 &= -\imath\partial_\phi.
\end{aligned}
\tag{31}
$$

These obey the commutation relations

$$
\begin{aligned}
[J_l, J_m] &= \imath\,\epsilon_{lmn}\,J_n, \\
[K_l, K_m] &= \imath\,\epsilon_{lmn}\,K_n, \\
[J_l, K_m] &= 0,
\end{aligned}
\tag{32}
$$

and J_i is obtained from K_i by interchanging $\phi \leftrightarrow \psi$. These operators are similar to those found in Ref. [12], which were deduced from the corresponding Infeld-Hull factorization. The Casimir operator J^2 is

$$
\begin{aligned}
4a^2J^2 = &-\partial_\theta^2 + a^2\left(-\partial_\phi^2 - \partial_\psi^2 + 2\partial_\phi\partial_\psi - \frac{1}{4}\right)\csc^2 a\theta \\
&+ a^2\left(-\partial_\phi^2 - \partial_\psi^2 - 2\partial_\phi\partial_\psi - \frac{1}{4}\right)\sec^2 a\theta - a^2.
\end{aligned} \tag{33}
$$

The other Casimir operator K^2 is identical. One may express the eigenfunction equation for a unitary representation of the group $SU(2)$ as

$$
\begin{aligned}
J^2|klm\rangle &= k(k+1)|klm\rangle,\ k = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \\
J_3|klm\rangle &= n|klm\rangle,\ n = -k, \ldots, 0, \ldots, k.
\end{aligned} \tag{34}
$$

If one chooses the eigenfunction $|klm\rangle = u^k_{mn}(\theta)\mathrm{e}^{\imath(\mathrm{l}\phi+\mathrm{m}\psi)}$ then Eqn. 34 becomes

$$
\begin{aligned}
&\frac{2a^2}{\mu}\ J\ {}^2u^k_{lm}(\theta) \\
&= \left[-\frac{1}{2\mu}\partial_\theta^2 + \frac{a^2}{2\mu}\left((l-m)^2 - \frac{1}{4}\right)\csc^2 a\theta + \frac{a^2}{2\mu}\left((l+m)^2 - \frac{1}{4}\right)\sec^2 a\theta - \frac{a^2}{2\mu}\right]u^k_{lm}(a\theta) \\
&= \frac{2a^2}{\mu}k(k+1)u^k_{lm}(\theta).
\end{aligned} \tag{35}
$$

This is the Schroedinger equation for the Poschl-Teller potential, which if we define the coefficients in the potential $A \equiv \hbar^2\gamma(\gamma-1)$ and $B \equiv \hbar^2\delta(\delta-1)$, gives $\gamma = l - m + \frac{1}{2}$, $\delta = l+m+\frac{1}{2}$ and $E_k = \frac{2a^2\hbar^2}{\mu}(k+\frac{1}{2})^2$. Since $l = k-j,\ j = 0, 1, \ldots, 2k$ the energy eigenvalues are

$$
E_k = \frac{a^2\hbar^2}{2\mu}(\gamma+\delta+2j)^2 \tag{36}
$$

with $j \geq \frac{1}{2}(1-\gamma-\delta)$. The same procedure for the K_i operators gives the same energy eigenvalues.

If one transforms the $SU(2)$ generators J_i, in Eqn. 31, into the corresponding ones for the Rosen-Morse I potential, using Eqn. 28, one obtains

$$
\begin{aligned}
J_1^{\mathrm{RM}} &= \imath\left(\frac{-1}{a}\cosh a\theta\sin\psi\partial_\theta - \cosh a\theta\cos\psi\partial_\phi + \sinh a\theta\cos\psi\partial_\psi\right), \\
J_2^{\mathrm{RM}} &= \imath\left(\frac{1}{a}\cosh a\theta\cos\psi\partial_\theta - \cosh a\theta\sin\psi\partial_\phi + \sinh a\theta\sin\psi\partial_\psi\right), \\
J_3^{\mathrm{RM}} &= -\imath\partial_\psi.
\end{aligned} \tag{37}
$$

The Casimir operator acting on the state $|klm\rangle \equiv u^k_{lm}(\theta)\mathrm{e}^{\imath(\mathrm{l}\phi-\mathrm{m}\psi)}$ gives

$$J^2 u^k_{lm}(\theta) = \left[\frac{-\cosh^2 a\theta}{a}\partial^2_\theta + (l^2+m^2)\cosh^2 a\theta + 2lm\sinh a\theta\cosh a\theta\right] u^k_{lm}(\theta) \tag{38}$$
$$= k(k+1)u^k_{lm}(\theta)$$

and $J_3^{\mathrm{RM}}|klm\rangle = -n|klm\rangle$. Multiplying by $-a^2\hbar^2\ \mathrm{sech}^2\mathrm{a}\theta/2\mu$ leads to the Schroedinger equation for the Rosen-Morse potential

$$\hbar^2\left[-\frac{1}{2\mu}\partial^2_\theta + \frac{a^2 lm}{\mu}\tanh a\theta - \frac{a^2k(k+1)}{2\mu}\ \mathrm{sech}^2\mathrm{a}\theta\right] u^k_{lm}(\theta) = -\frac{a^2\hbar^2(l^2+m^2)}{2\mu}u^k_{lm}(\theta) \tag{39}$$

with parameters $A = a^2lm/\mu$ and $B = a^2k(k+1)/2\mu$ and energy eigenvalue $E = -a^2\hbar^2(l^2+m^2)/2\mu$. Since the energy eigenvalues are non-positive only the bound states energies may be found. Again for a unitary representation of $SU(2)$ we have $-m = -k+j,\ j = 0,1,\ldots,2k$. Substituting this in the equation for the energy eigenvalue and expressing the result in terms of the potential coefficients

$$E_j = -\hbar^2\left[\frac{\mu A^2}{2a^2}\left(\frac{1}{n^2}\right) + \frac{a^2}{2\mu}n^2\right] \tag{40}$$
$$n = -\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{8\mu B}{a^2}} - j,\ j = 0,1,\ldots,\left(-1+\sqrt{1+\frac{8\mu B}{a^2}}\right).$$

Furthermore we may assume that $A \geq 0$, since under the change of variables $\theta \to -\theta$, $A \to -A$. Similar to the Poschl-Teller case, the other $SU(2)$ operators K_i may be found from J_i by exchanging $\phi \leftrightarrow \psi$ and furthermore the Casimirs are equal, $K^2 = J^2$. Therefore, with $K_3|klm\rangle = l|klm\rangle$ the range of the eigenvalue is $l = -k, -k+1, \ldots, k-1, k$ and one finds the following bound on the coefficients in the potential in order for the existence of a bound state

$$\left(\frac{\mu A}{a^2}\right)^{\frac{1}{2}} = lm \leq k^2 = \left(-\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{8\mu B}{a^2}}\right)^2. \tag{41}$$

V. CONCLUSION

We have shown that if a particular type of operator transformation, which is not necessarily unitary, exists between two Schroedinger operators there is a procedure for finding an

algebraic relation between the respective propagators and that the two eigenvalue problems have the same formulation in terms of Lie group generators. Also a knowledge of the Fourier transform of the propagator for the new potential allows one, in principle, to find the energy eigenvalues and wavefunctions for both the bound and scattering states. One interesting generalization of this procedure would be to find such operator transformations between multiparticle exactly solvable systems, such as those of the Calogero-Sutherland type.

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Potential Name	Potential $V(x)$	Transformation Function $f(x)$	Propagator $G(x_f, x_0, E)$
Radial Harmonic Oscillator	$\frac{\hbar^2 l(l+1)}{2\mu x^2} + \frac{1}{2}\mu\omega x^2,\ x > 0$	—	$G_{RHO}\left(x_f,\ x_0,\ E;\ l,\ \frac{1}{2}\mu\omega^2\right)$
Coulomb	$\frac{\hbar^2 l(l+1)}{2\mu x^2} - \frac{e^2}{x},\ x > 0$	x^2	$2\,(x_0 x_f)^{\frac{1}{4}}\, G_{RHO}\left(x_f^{\frac{1}{2}},\ x_0^{\frac{1}{2}},\ 4e^2;\ 2l + \frac{1}{2},\ -4E\right)$
Morse	$Ae^{-2ax} - Be^{-ax}$	$-\frac{2}{a}\ln x$	$\frac{-2}{a}\, e^{\frac{1}{4}a(x_0+x_f)}\, G_{RHO}\left(e^{-\frac{1}{2}ax_f},\ e^{-\frac{1}{2}ax_0},\ \frac{4B}{a^2};\ -\frac{1}{2} + \frac{2\sqrt{-2\mu E}}{\hbar a},\ \frac{4A}{a^2}\right)$
Trig. Poschl-Teller	$A\csc^2 ax + B\sec^2 ax,$ $0 < x < \frac{\pi}{2}$	—	$G_{\text{trig PT}}\left(x_f,\ x_0,\ E;\ A,\ B\right)$
Trig. Scarf	$A\csc^2 ax + B\cot ax\csc ax,$ $0 < x < \pi$	$2r$	$2\,G_{\text{trig PT}}\left(\frac{1}{2}x_f,\ \frac{1}{2}x_0,\ 4E;\ A+B,\ A-B\right)$
Rosen-Morse I	$A\tanh ax - B\ \operatorname{sech}^2 ax$	$\frac{1}{a}\ \operatorname{arctanh}(\cos 2ar)$	$2\,(\cosh ax_f\ \cosh ax_0)^{\frac{1}{2}}\, G_{\text{trig PT}}\Big(\frac{1}{2a}\arccos\tanh ax_f,$ $\frac{1}{2a}\arccos\tanh ax_0,\ -4B + \frac{\hbar^2 a^2}{2};\ A - E - \frac{\hbar^2 a^2}{8\mu},\ -A - E - \frac{\hbar^2 a^2}{8\mu}\Big)$
Rosen-Morse II	$A\ \operatorname{csch}^2 ax - B\coth ax\ \operatorname{csch} ax$	$\frac{1}{a}\ \operatorname{arccosh}\left(\frac{\cos 2ar+3}{1-\cos 2ar}\right)$	$2\ \operatorname{sech}\frac{x_f}{2}\ \operatorname{sech}\frac{x_0}{2}\, G_{\text{trig PT}}\Big(\frac{1}{2a}\arccos\left(\frac{\cosh ax_f - 3}{\cosh ax_f + 1}\right),$ $\frac{1}{2a}\arccos\left(\frac{\cosh ax_0 - 3}{\cosh ax_0 + 1}\right),\ \frac{1}{8\mu}\hbar^2 a^2 + A + B;\ -\frac{1}{8\mu}\hbar^2 a^2 - 4E,\ A - 2B\Big)$
Eckart	$-A\coth ax + B\ \operatorname{csch}^2 ax$	$\frac{1}{a}\ \operatorname{arccoth}\left(\frac{\cos 2ar+3}{1-\cos 2ar}\right)$	$\sqrt{2}\,(\coth ax_f - 1)^{-\frac{1}{4}}\,(\coth ax_0 - 1)^{-\frac{1}{4}}\, G_{\text{trig PT}}\Big(\frac{1}{2a}\arccos\left(\frac{\coth ax_f - 3}{\coth ax_f + 1}\right),$ $\frac{1}{2a}\arccos\left(\frac{\coth ax_0 - 3}{\coth ax_0 + 1}\right),\ E - A;\ 4B + \frac{3\hbar^2 a^2}{8\mu},\ -E - A - \frac{\hbar^2 a^2}{8\mu}\Big)$
Hyp. Poschl-Teller	$A\ \operatorname{csch}^2 ax - B\ \operatorname{sech}^2 ax$	—	$G_{\text{hyp PT}}\left(x_f, x_0, E; A, B\right)$
Hyp. Scarf	$A\ \operatorname{csch}^2 ax + B\coth ax\ \operatorname{csch} ax$	$2r$	$2\,G_{\text{hyp PT}}\left(\frac{1}{2}x_f,\ \frac{1}{2}x_0,\ 4E;\ A+B,\ A-B\right)$

Table 1: The shape invariant potentials are shown with the function $f(x)$ of Eqn. 26 used to transform the Schroedinger operator to either the radial harmonic oscillator, trigonometric Poschl-Teller, or hyperbolic Poschl-Teller potential. The propagator for each potential, defined in Eqn. 2, is also given as a function of the propagator for one of these three potentials.